where the terms of matrix $[A]$ are functions of the coordinates x, y, and z By using the strain displacement relations, one can write the strain distribution as

$$\{\epsilon\} = [W]\{\alpha\} \tag{2}$$

and when the stress-strain relation

$$\{\sigma\} = [E]\{\epsilon\} \tag{3}$$

is introduced, one can express the internal strain energy as

$$U = \frac{1}{2}\lfloor\alpha\rfloor[G]\{\alpha\} \tag{4}$$

where

$$[G] = \int_V [W^T][E][W]dV \tag{5}$$

From Eq (1), one can also express the n generalized displacements $\{q\}$ at the node points in terms of the undetermined coefficients $\{\alpha\}$:

$$\underset{n\times 1}{\{q\}} = \underset{n\times(n+l)}{[B]} \underset{(n+l)\times 1}{\{\alpha\}} \tag{6}$$

By partitioning the matrix $[B]$ one can write

$$\underset{n\times 1}{\{q\}} = \underset{n\times n}{[B\ ]}\underset{n\times 1}{\{\alpha_a\}} + \underset{u\times l}{[B_b]}\underset{l\times 1}{\{\alpha_b\}} \tag{7}$$

One can solve for $\{\alpha_a\}$ in terms of $[q]$ and $\{\alpha_b\}$ and write

$$\{\alpha\} = \begin{Bmatrix}\alpha_a \\ \alpha_b\end{Bmatrix} = \begin{bmatrix} \underset{(n\times n)}{B_a^{-1}} & \underset{(n\times l)}{-B_a^{-1}B_b} \\ \underset{(l\times n)}{0} & \underset{(l\times l)}{I}\end{bmatrix}\begin{Bmatrix} q \\ \alpha_b\end{Bmatrix} \tag{8}$$

Let

$$[M] = \begin{bmatrix} B_a^{-1} & -B_a^{-1}B_b \\ 0 & I\end{bmatrix} \tag{9}$$

The strain energy U can thus be written as

$$U = \frac{1}{2}\lfloor q,\alpha_b\rfloor[K]\begin{Bmatrix} q \\ \alpha_b\end{Bmatrix} \tag{10}$$

where

$$[K] = [M^T][G][M] \tag{11}$$

The total potential energy including the work done by the generalized forces $\{Q\}$ is

$$\pi_p = U - [q]\{Q\} \tag{12}$$

The condition of minimum potential energy [i e, $\partial\pi_p/\partial q_i = 0\ (i = 1, \quad n)$, and $\partial\pi_p/\partial\alpha_{bj} = 0 (j = 1 \quad l)$] yields

$$\begin{bmatrix} \underset{(n\times n)}{K_{aa}} & \underset{(n\times l)}{K_{ab}} \\ \underset{(l\times n)}{K_b} & \underset{(l\times l)}{K_{bb}}\end{bmatrix}\begin{Bmatrix} q \\ \alpha_b\end{Bmatrix} = \begin{Bmatrix} Q \\ 0\end{Bmatrix} \tag{13}$$

in which the $[K]$ matrix has been partitioned It is seen that $\{\alpha_b\}$ can be expressed in terms of $\{q\}$ by solving the last l equations, and, after eliminating $\{\alpha_b\}$, the following equation results:

$$([K_{aa}] - [K_{ab}][K_{bb}^{-1}][K_b\ ])\{q\} = \{Q\} \tag{14}$$

By definition, the element stiffness matrix is

$$[k] = [K_{aa}] - [K_{ab}][K_{bb}^{-1}][K_b\ ] \tag{15}$$

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Chemical Kinetic Analysis of Rocket Exhaust Temperature Measurements

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RECENT investigations[1–3] of nozzle performance and flow conditions with finite-rate chemical reactions indicate that, if the reaction paths and rate constants are known, then the flow parameters can be predicted to a significantly higher degree of accuracy than by thermodynamic methods alone Conversely, it is sometimes possible to infer approximate values of the rate constants of the controlling reactions for a particular hypothesized mechanism from a comparison of theoretical predictions with experimental flow parameters Studies of this nature have been performed by Franciscus and Lezberg[4] and by Hoglund, Byron, and Carlson[5] There is a considerable difference between the values of rate constants which these two groups found best to fit their respective experimental data The differences in rate constants, under some conditions, lead to significant differences in predicted exhaust conditions

During the past few years, a technique of studying the infrared emissivities of gases, using a small rocket motor with a contoured nozzle to generate the hot gases, has been developed by Ferriso and his co-workers[6 7] In the course of these studies the exhaust temperatures of the rocket, operated with different propellants and various mixture ratios, were measured by an infrared emission-absorption technique[8] Temperatures between 530° and 2450°K have been measured by this method to an estimated accuracy of 50°K The rocket "burner" was of nominal 150-lb thrust, water-cooled, and of apparently high combustion efficiency (>95% theoretical C^*) The nozzle was designed by the Foelsch method[9] to produce a homogeneous, axially directed exhaust jet at 1 atm static pressure Photographs and shadowgraphs[10] of the exhaust show no shocks if the exit pressure and ambient pressure are balanced (by adjusting the propellant flow rate) Measurements are made less than 2 mm downstream of the exit plane The homogeneity of the gas sample at this region is shown by the constancy of measured total pressure across the exit plane and by the good agreement between gaseous spectral emissivities measured under these conditions and in absorption cells

Exhaust temperatures of this small motor, operated with RP-1 and gaseous O_2 at mixture ratios from 2 2 to 6, were measured under balanced conditions Measured temperatures fall between the values calculated for equilibrium and frozen flow The measured and thermodynamically calculated temperatures (solid lines) are shown in Fig 1

Figure 1 also shows values of the exhaust temperature calculated by applying a sudden-freezing model to the recombination reactions Analyses indicate that the energy release attendant upon the recombinations dominates the total energy release in the expanding flow and that other reactions are, to a good approximation, thermally insignificant The reaction scheme is the same used by Hoglund et al Three different sets of rate constants were employed: 1) those suggested by Hoglund et al from flame data; 2) those selected by Franciscus and Lezberg from a combination of shock-tube work and flame data; and 3) those representing the "lower limit" of published values

Received December 27, 1963 The author wishes to express his thanks to C C Ferriso and C B Ludwig for several helpful discussions of their data and for the use of those data here This work was supported by the Advanced Research Projects Agency through the U S Air Force Space Systems Division, Contract No AF 04(695)-140

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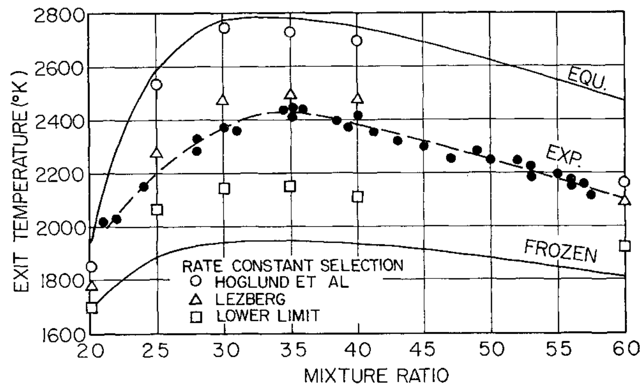

Fig 1 Comparison of experimental and theoretical exit temperatures ($RP\text{-}1/GO_2$, ϵ = 5 23:1, P_E = 1 atm)

Since the motor is water-cooled, the exhaust temperature is presumably affected by heat losses from the combustion chamber and nozzle The heat losses for this particular engine were not measured However, losses from a geometrically identical engine burning hydrogen and oxygen as propellants were measured by monitoring cooling water temperature and were found to agree within 20% with those calculated by the method of Bartz [11] Calculations of exit temperatures for the H_2/O_2 engine with the measured heat losses,[7] assuming that all heat is lost from the combustion chamber, lead to a decrease of about 50°K in exit temperature near the stoichiometric mixture ratio for the same operating conditions and freezing pressure

A quick estimate of the heat losses for a stoichiometric $RP\text{-}1/O_2$ system indicates that they are no greater than those for the H_2/O_2 system Since heat losses from regions of the nozzle downstream of the chamber produce a larger effect on exit temperature than equal losses from the chamber, it is expected that the temperatures that were measured on this $RP\text{-}1/O_2$ engine were as much as 100° to 150°K lower than the temperatures that we would have measured if no heat transfer had occurred

The results shown in Fig 1 indicate that, if the reaction scheme proposed by Hoglund et al is valid, then the rate constants suggested by Franciscus and Lezberg give much better agreement with the measured exit temperatures than do either those of Hoglund et al or the "lower limit" values The important reactions and rate constants are

$$H + H + M \rightleftarrows H_2 + M \qquad k_1 = 3 \times 10^{18}\, T^{-1} \quad (1)$$

$$H + OH + M \rightleftarrows H_2O + M \qquad k_2 = 7\ 5 \times 10^{19}\, T^{-1} \quad (2)$$

$$H + O + M \rightleftarrows OH + M \qquad k_3 \approx 10^{15} - 10^{16} \quad (3)$$

where k's are in $cm^6/mole^2$-sec Since k_2 is thought to be considerably larger than k_1 or k_3, and since much more OH than either H or O is present near the stoichiometric mixture ratio, the results of the present investigation should be regarded as an approximate determination of the magnitude of k_2 and as confirmation of the hypothesis that $k_2 \geq 10k_1$ Since a "sudden-freezing" analysis was used in this work and in that of Franciscus and Lezberg, the value of k_2 determined here should be employed only in "sudden-freezing" analyses and not in more detailed calculations [1] Better determination of k_1 could be obtained by running the motor, with hydrogen as fuel, extremely rich, or by an entirely different technique (Many determinations of k_1 by shock-tube techniques have appeared in the literature) A test of the proposed mechanism, which neglects HO_2 as a participant, could be made by running the motor very lean Some runs of this sort have been made but have not been analyzed

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Transpiration and Film Cooling Combined with External Cooling

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Nomenclature

b = duct circumference
c_p = specific heat
d = wall thickness
h = convective heat-transfer coefficient
k = thermal conductivity
l = cooled duct length
q = heat flux
w = injectant flow rate per unit length of duct circumference
C = "effectiveness" of convective coolant
R = effectiveness of transpiration coolant
Re = Reynolds number of film injection based on slot width
S = slot width
T = temperature
V = velocity
W = total injection rate
η = effectiveness of film coolant
λ = ratio of required injection rates
μ = dynamic viscosity
ρ = density

Subscripts

a = allowable wall temperature
ad = adiabatic
c = coolant
g = gas side
w = wall

Superscript

0 = heat-transfer data with no injection

Received December 31, 1963
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